Airfoil Shape Optimization Using Output-Based Adapted Meshes

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Outline

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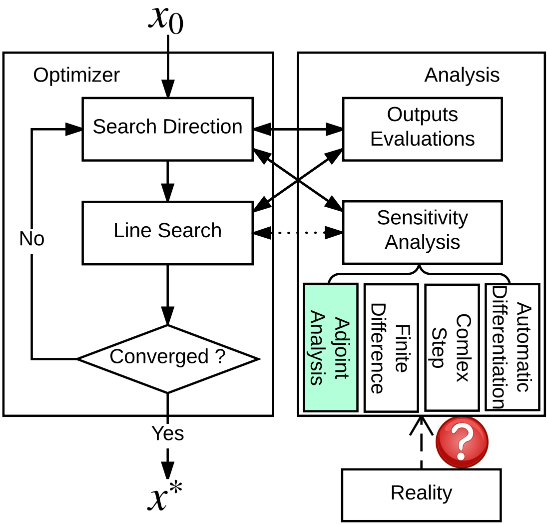
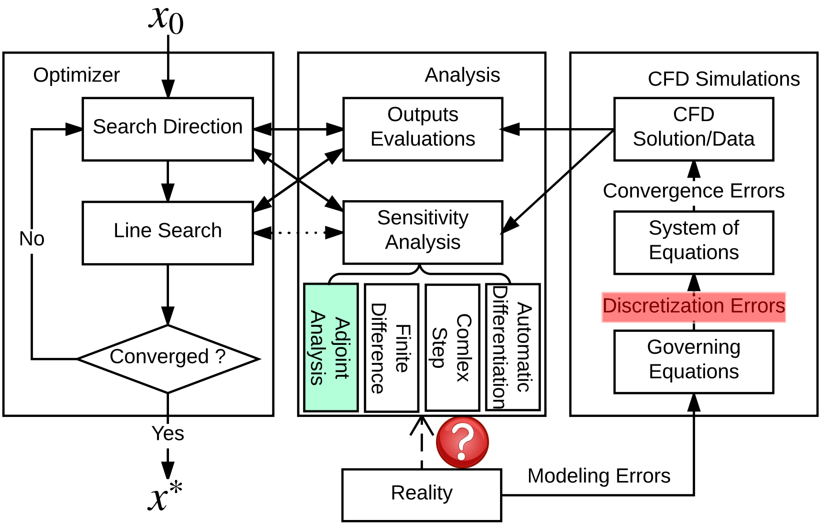
Design/Optimization: Numerical Optimization + CFD Analysis



Aerodynamic Shape Design/Optimization

Design/Optimization: Numerical Optimization + CFD Analysis



Improving Optimization Accuracy and Efficiency

Traditional methods

- a *priori* mesh: numerical error not controlled during optimization
- fixed fidelity: optimizing on fixed mesh resolution

Proposed method

- Multi-fidelity optimization: reduce the computational resources at the early stages of optimization
- Adjoint based error estimation and mesh adaptation: actively control the numerical error during the optimization
- $\xRightarrow{\textit{Integration}}$ Multi-fidelity optimization driven by error estimation and mesh adaptation:
  Initial shape $\Rightarrow$ Optimal design, Coarse mesh $\Rightarrow$ Fine mesh
  Goal: prevent over-refining and over-optimizing

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1. Introduction
2. Optimization Problem
3. Discretization
4. Error Estimation and Mesh Adaptation
5. Optimization Approach
6. Results and Discussion
7. Conclusions and Future Work

Optimization Problem Formulation

General optimization problem

- Determine the design variables $\mathbf{x}$ that minimize the objective function J:

$$\min_{\mathbf{x}} \; J(\mathbf{U};\mathbf{x})$$
$$\text{s.t.} \quad \mathbf{R}^{e}(\mathbf{U};\mathbf{x}) = \mathbf{0}$$
$$\mathbf{R}^{ie}(\mathbf{U};\mathbf{x}) \quad \mathbf{0}$$

- $\mathbf{U}$ denotes the flow variables, $\mathbf{R}^{e}$ and $\mathbf{R}^{ie}$ are the equality and inequality constraints.

Aerodynamic optimization

- Objective and constraints are aerodynamic outputs
- Physical feasibility: Flow variables $\mathbf{U}$ are solved within a feasible design space to satisfy the flow equations,

$$\mathbf{R}(\mathbf{U};\mathbf{x}) = \mathbf{0}; \quad 8 \; \mathbf{x} \; 2$$

Adjoint and Design Equations

Objective and trim constraints

- Objective outputs: directly targeted for mesh adaptation, denoted as J^{adapt}
- Trim constraints: We only consider the equality constraints $\mathbf{R}^{\text{e}}$ and active inequality constraints $\mathbf{R}_{\text{a}}^{\text{ie}}$,

$$\mathbf{R}^{\text{trim}} = [\mathbf{R}^{\text{e}}\ \mathbf{R}_{\text{a}}^{\text{ie}}]^T = \mathbf{J}^{\text{trim}} \qquad \mathbf{J}^{\text{trim}} = \mathbf{0}$$

Augmented Lagrangian functions

The adjoint-based optimization is equivalent to searching for the stationary point of the augmented Lagrangian function,

$$L(\mathbf{U}; \mathbf{x};\ \ ;\ \ ) = J^{\text{adapt}}(\mathbf{U}; \mathbf{x}) + \ {}^T\mathbf{R}(\mathbf{U}; \mathbf{x}) + \ {}^T\mathbf{R}^{\text{trim}}(\mathbf{U}; \mathbf{x})$$

where and are Lagrange multipliers associated with the flow equations and the trim constraints, respectively

Optimality Condition

- First-order optimality (Karush-Kuhn-Tucker) condition

$$\frac{\partial L}{\partial \mathbf{x}} = \frac{\partial J^{\text{adapt}}}{\partial \mathbf{x}} + {}^{T}\frac{\partial \mathbf{R}}{\partial \mathbf{x}} + {}^{T}\frac{\partial \mathbf{R}^{\text{trim}}}{\partial \mathbf{x}} = \mathbf{0} \qquad \text{optimal design}$$

$$\frac{\partial L}{\partial \mathbf{U}} = \frac{\partial J^{\text{adapt}}}{\partial \mathbf{U}} + {}^{T}\frac{\partial \mathbf{R}}{\partial \mathbf{U}} + {}^{T}\frac{\partial \mathbf{R}^{\text{trim}}}{\partial \mathbf{U}} = \mathbf{0} \qquad \text{coupled adjoint}$$

$$\frac{\partial L}{\partial} = \mathbf{R}(\mathbf{U}; \mathbf{x}) = \mathbf{0} \qquad \text{physics feasibility}$$

$$\frac{\partial L}{\partial} = \mathbf{R}^{\text{trim}}(\mathbf{U}; \mathbf{x}) = \mathbf{0} \qquad \text{trim condition}$$

- Always physically feasible: $\mathbf{R}(\mathbf{U}; \mathbf{x}) = \mathbf{0}$
- Choose coupled adjoints variables, such that,

$$\frac{\partial L}{\partial \mathbf{U}} = \mathbf{0} \Rightarrow \quad {}^{T} = \quad \frac{\partial J^{\text{adapt}}}{\partial \mathbf{U}} + {}^{T}\frac{\partial \mathbf{R}^{\text{trim}}}{\partial \mathbf{U}} \quad \frac{\partial \mathbf{R}}{\partial \mathbf{U}}^{1}$$

$$= (\;{}^{\text{adapt}} + \;{}^{\text{trim}} \;)^{T}$$

Reduced optimality condition

- Coupled adjoints: $\ ^T = (\ ^{\text{adapt}} + \ ^{\text{trim}})^T$, where

$$\frac{\partial \mathbf{R}}{\partial \mathbf{U}}^T \ ^{\text{adapt}} + \frac{\partial J^{\text{adapt}}}{\partial \mathbf{U}}^T = \mathbf{0}; \qquad \frac{\partial \mathbf{R}}{\partial \mathbf{U}}^T \ ^{\text{trim}} + \frac{\partial \mathbf{J}^{\text{trim}}}{\partial \mathbf{U}}^T = \mathbf{0}$$

- Sensitivity Analysis

$$\begin{aligned} \frac{\partial L}{\partial \mathbf{x}} &= \frac{\partial J^{\text{adapt}}}{\partial \mathbf{x}} + \ ^T \frac{\partial \mathbf{R}}{\partial \mathbf{x}} + \ ^T \frac{\partial \mathbf{R}^{\text{trim}}}{\partial \mathbf{x}} \\ &= \frac{\partial J^{\text{adapt}}}{\partial \mathbf{x}} + (\ ^{\text{adapt}})^T \frac{\partial \mathbf{R}}{\partial \mathbf{x}} + \ ^T \frac{\partial \mathbf{R}^{\text{trim}}}{\partial \mathbf{x}} + (\ ^{\text{trim}})^T \frac{\partial \mathbf{R}}{\partial \mathbf{x}} \\ &= \frac{dJ^{\text{adapt}}}{d\mathbf{x}} + \ ^T \frac{d\mathbf{J}^{\text{trim}}}{d\mathbf{x}} \end{aligned}$$

- Reduced optimality condition:

$$\begin{aligned} \frac{\partial L}{\partial \mathbf{x}} &= \frac{dJ^{\text{adapt}}}{d\mathbf{x}} + \ ^T \frac{d\mathbf{J}^{\text{trim}}}{d\mathbf{x}} = \mathbf{0} \\ \frac{\partial L}{\partial \ } &= \mathbf{R}^{\text{trim}} = \mathbf{0} \end{aligned}$$

Outline

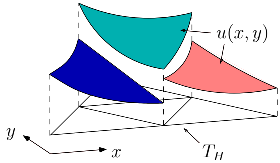
Discretization

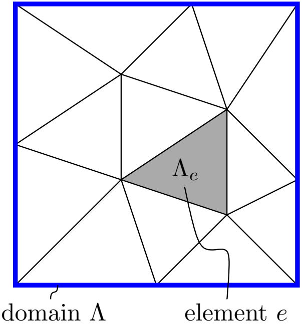
- Conservation law: $\partial_t \mathbf{u} + \nabla \cdot \vec{\mathbf{H}}(\mathbf{u}, \nabla \mathbf{u}) = \mathbf{0}$

  where $\underbrace{\vec{\mathbf{H}}}_{\text{total flux}} = \underbrace{\vec{\mathbf{F}}(\mathbf{u})}_{\text{inviscid flux}} + \underbrace{\vec{\mathbf{G}}(\mathbf{u}, \nabla \mathbf{u})}_{\text{viscous flux}}$

- DG approx of order p_e on each element:

$$\mathbf{u}_h(\vec{x}) = \sum_{e=1}^{N_e} \sum_{n=1}^{N_{p_e}} \mathbf{U}_{e,n} \phi_{e,n}(\vec{x})$$





N_e = # of elements

p_e = approx order on element e

N_{p_e} = # of basis fcns on element e

$\phi_{e,n}(\vec{x})$ = n^{th} basis fcn of order p_e on e

$\mathbf{U}_{e,n}$ = coefficients vector of n^{th} basis function on element e

Outline

Output-Based Error Estimation

For a given configuration (design) $\mathbf{x}$, the outputs (objective and constraints) are based on a **pure** CFD flow solve.

Output Error: $J = J_H(\mathbf{U}_H; \mathbf{x}) \quad J(\mathbf{U}; \mathbf{x})$

This is the difference between J computed with the discrete system solution, $\mathbf{U}_H$, and J computed with the *exact* solution, $\mathbf{U}$.

Error Surrogate: $J = J_H(\mathbf{U}_H; \mathbf{x}) \quad J_h(\mathbf{U}_h; \mathbf{x})$

The difference between outputs on coarse and fine discretizations.

$$\text{coarse space:} \quad \rightarrow \underbrace{\mathbf{R}_H(\mathbf{U}_H; \mathbf{x}) = \mathbf{0}}_{N_H \text{ flow equations}} \rightarrow \underbrace{\mathbf{U}_H}_{\text{state} \in \mathbb{R}^{N_H}} \rightarrow J_H(\mathbf{U}_H; \mathbf{x})$$

$$\text{fine space:} \quad \rightarrow \underbrace{\mathbf{R}_h(\mathbf{U}_h; \mathbf{x}) = \mathbf{0}}_{N_h \text{ flow equations}} \rightarrow \underbrace{\mathbf{U}_h}_{\text{state} \in \mathbb{R}^{N_h}} \rightarrow J_h(\mathbf{U}_h; \mathbf{x})$$

Adjoint-based Error Estimation

- State injection: $\mathbf{U}_h^H = \mathbf{I}_h^H \mathbf{U}_H$
- $\mathbf{U}_h^H$ will generally not satisfy the fine-space equations,

$$\mathbf{R}_h(\mathbf{U}_h^H; \mathbf{x}) \neq \mathbf{0}$$

- Recall the definition of the output adjoint, $\frac{\partial \mathbf{R}}{\partial \mathbf{U}}^T \quad + \frac{\partial J}{\partial \mathbf{U}}^T = \mathbf{0}.$ relates the residual perturbation to an output perturbation,

$$J = \underbrace{\frac{\partial J}{\partial \mathbf{U}} \mathbf{U} = \quad {}^T \frac{\partial \mathbf{R}}{\partial \mathbf{U}} \mathbf{U}}_{\text{adjoint definition}} \qquad \overbrace{\qquad {}^T \mathbf{R}}^{\text{residual linearization}}$$

$$J = J_H(\mathbf{U}_H; \mathbf{x}) \quad J_h(\mathbf{U}_h; \mathbf{x}) = J_h(\mathbf{U}_h^H; \mathbf{x}) \quad J_h(\mathbf{U}_h; \mathbf{x})$$

$$\quad {}_h^T \mathbf{R}_h = \quad {}_h^T \mathbf{R}_h(\mathbf{U}_h^H; \mathbf{x})$$

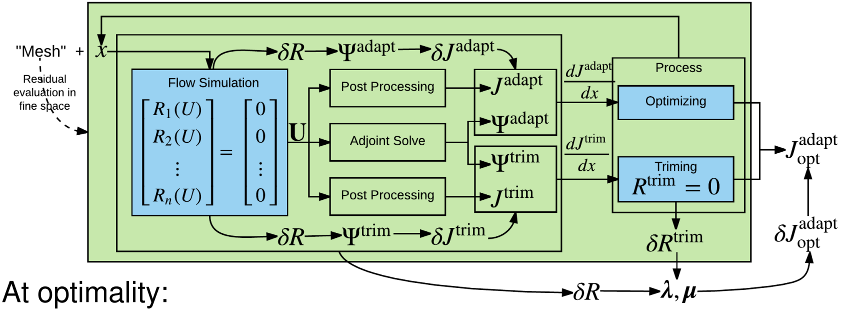
- Error (Adapt) indicator: the error is localized in each element and serves as an adaptation indicator, $\quad_e = | \quad {}_{h;e}^T \mathbf{R}_{h;e}(\mathbf{U}_h^H; \mathbf{x})|$

Error Estimation and Mesh Adaptation for Optimization

- Numerical error affects both objective and constraint outputs.
- How to estimate the error and adapt the mesh efficiently?
- Both errors can be obtained via adjoints.
- Possible adaptation strategies:
  1. Adapt only on the objective
     Error due to inexact constraints satisfaction
  2. Adapt equally on the objective and constraints
     Inefficient, expensive to keep all outputs very accurate
  3. Adapt on combined/weighted outputs
     The weights? Adapt more on objective/constraints?
  4. Adapt on the optimization problem (coupled adjoint)

coarse space: $\mathbf{x}_0 \rightarrow$ optimization $\rightarrow \underbrace{\mathbf{x}_H, \mathbf{U}_H}_{\text{optimal design}} \rightarrow J_H(\mathbf{U}_H, \mathbf{x}_H)$

fine space: $\mathbf{x}_0 \rightarrow$ optimization $\rightarrow \underbrace{\mathbf{x}_h, \mathbf{U}_h}_{\text{optimal design}} \rightarrow J_h(\mathbf{U}_h, \mathbf{x}_h)$



At optimality:

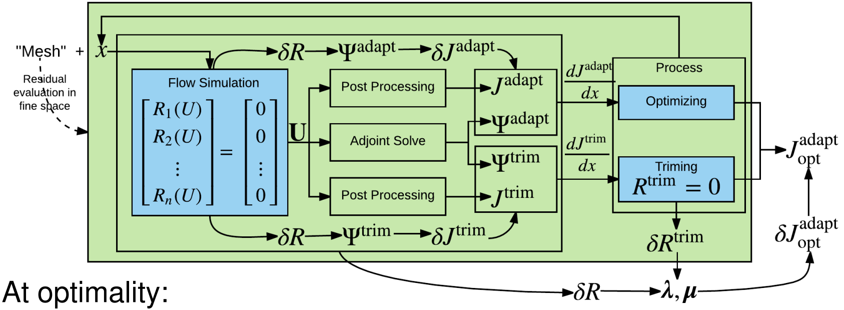
$$\frac{\partial L}{\partial \mathbf{x}} = \frac{\partial J^{adapt}}{\partial \mathbf{x}} + \lambda^T \frac{\partial \mathbf{R}}{\partial \mathbf{x}} + \mu^T \frac{\partial \mathbf{R}^{trim}}{\partial \mathbf{x}} = \mathbf{0}$$

$$\frac{\partial L}{\partial \mathbf{U}} = \frac{\partial J^{adapt}}{\partial \mathbf{U}} + \lambda^T \frac{\partial \mathbf{R}}{\partial \mathbf{U}} + \mu^T \frac{\partial \mathbf{R}^{trim}}{\partial \mathbf{U}} = \mathbf{0}$$

$$\frac{\partial L}{\partial \lambda} = \mathbf{R}(\mathbf{U}; \mathbf{x}) = \mathbf{0}$$

$$\frac{\partial L}{\partial \mu} = \mathbf{R}^{trim}(\mathbf{U}; \mathbf{x}) = \mathbf{0}$$

Error Estimation and Mesh Adaptation for Optimization



At optimality:

$$J^{adapt}_{opt} = \quad {}^T_h \; \mathbf{R}_h \qquad {}^T_h \; \mathbf{R}^{trim}_h$$

$$= \quad {}^T_h \mathbf{R}_h(\mathbf{U}^H_h; \mathbf{x}_H) \qquad {}^T_h \mathbf{R}^{trim}_h(\mathbf{U}^H_h; \mathbf{x}_H)$$

$$= \quad (\quad {}^{adapt}_h + \quad {}^{trim}_h \quad {}_h)^T \mathbf{R}_h(\mathbf{U}^H_h; \mathbf{x}_H) \qquad {}^T_h \underbrace{(\mathbf{J}^{trim}_h(\mathbf{U}^H_h; \mathbf{x}_H) \qquad \mathbf{J}^{trim})}_{\mathbf{0}}$$

$$= \quad J^{adapt}(\mathbf{x}_H) + \quad {}^T_h \quad \mathbf{J}^{trim}(\mathbf{x}_H)$$

Note: $\mathbf{J}^{trim}_h(\mathbf{U}^H_h; \mathbf{x}_H) = \mathbf{J}^{trim}_H(\mathbf{U}_H; \mathbf{x}_H) = \mathbf{J}^{trim} = \mathbf{J}^{trim}_h(\mathbf{U}_h; \mathbf{x}_h)$

Error Estimation and Mesh Adaptation for Optimization

Optimality error estimation

$$\frac{dJ^{\text{adapt}}}{d\mathbf{x}} + {}^{T}\frac{d\mathbf{J}^{\text{trim}}}{d\mathbf{x}} = \mathbf{0}$$

$$J^{\text{adapt}}_{\text{opt}} = \underbrace{J^{\text{adapt}}(\mathbf{x}_H)}_{\text{objective error only}} + \underbrace{{}^{T}_{h}\ \mathbf{J}^{\text{trim}}(\mathbf{x}_H)}_{\text{inexact constraints satisfaction}}$$

What does mean?

It is the objective sensitivity w.r.t constraints and measures how much the constraints error can affect the optimal objective.

Mesh adaptation implementation

Adapt (error) indicator: $= j \; {}^{T}_{h;} \; \mathbf{R}_{h;} \; (\mathbf{U}_h^H; \mathbf{x}_H) j$

Combined indicator: $_{;\text{opt}} = {}^{\text{adapt}} + j \;\; j^{T} \;\; {}^{\text{trim}}$

$_h$: reconstructing the coarse-space adjoints $_H$

$_h$: extracted from the optimizer on the coarse space

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Multi-fidelity Optimization with Error Control

Geometry and Mesh

- Airfoil parameterization: Hicks-Henne basis function
- Design parameters: airfoil shape + angle of attack
- Mesh movement: Radial Basis Function (RBF) interpolation

Optimization Algorithm

- Sequential Least Squares Programming (SLSQP)
- Broyden-Fletcher-Goldfarb-Shanno (BFGS) Hessian update
- Inexact line search: weak Wolfe condition

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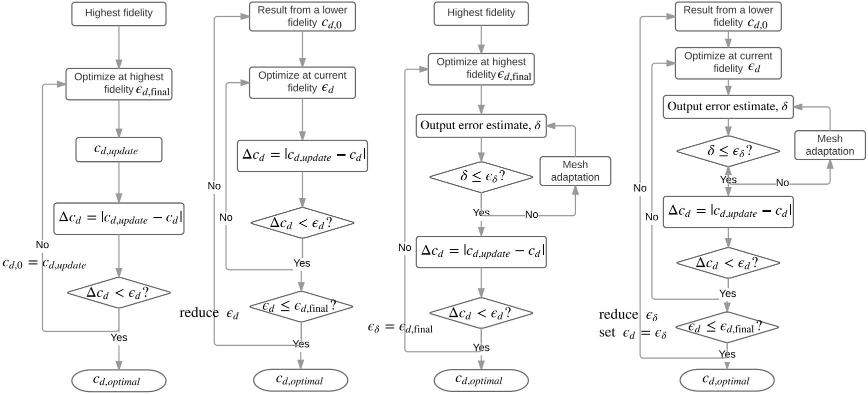

Outline

Laminar, Subsonic Flow (nearly feasible starting point)

NACA 0012, $Re = 5000$, $M_1 = 0{:}5$, $_0 = 0$
$J^{\text{adapt}} = c_d$, $J^{\text{trim}} = c_l$, $J^{\text{trim}} = 0{:}02$, $A \quad A_{min}$

Laminar, Subsonic Flow (nearly feasible starting point)

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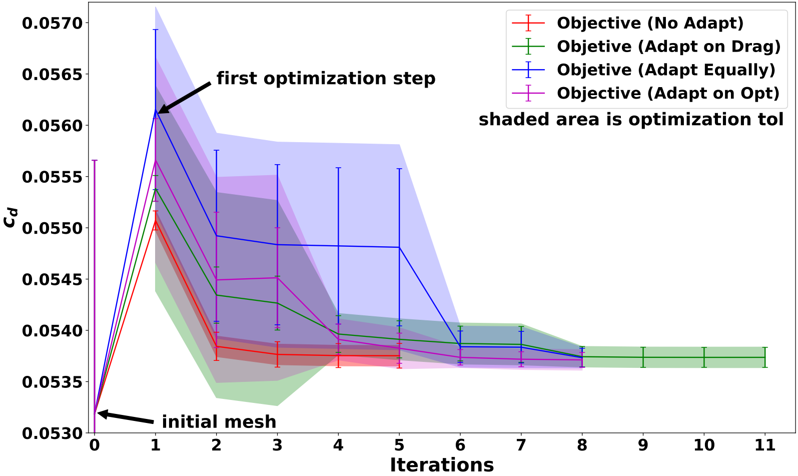

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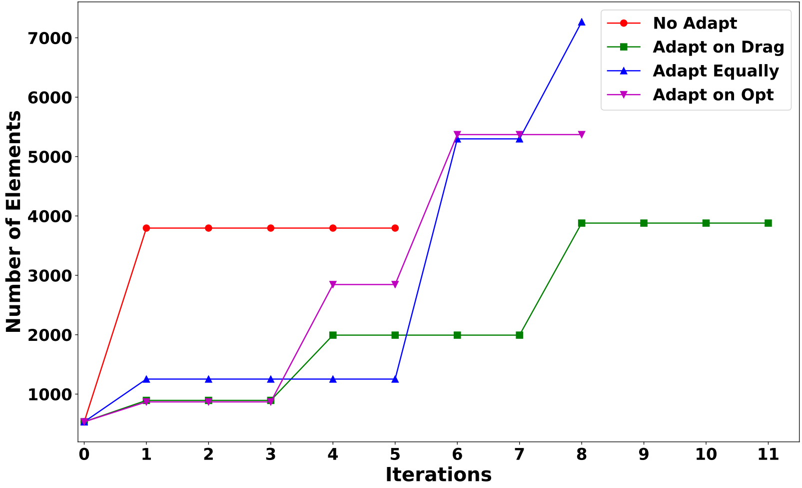
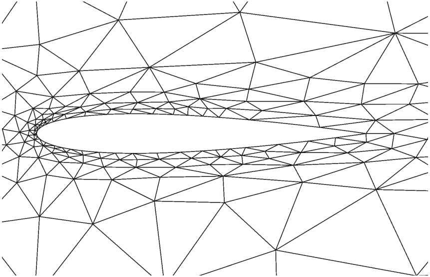
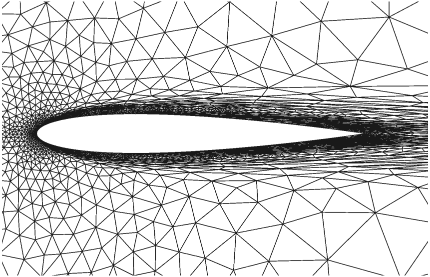


Mesh Evolution



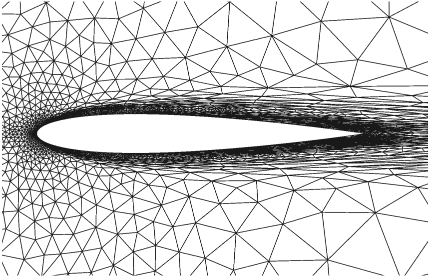
initial mesh (533 elements)

Mesh Evolution



no adapt (3796 elements)

Mesh Evolution



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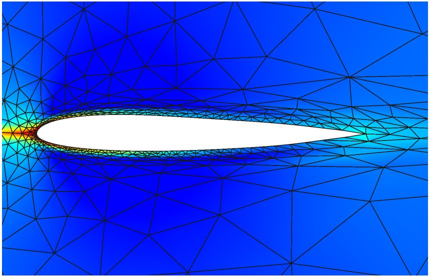
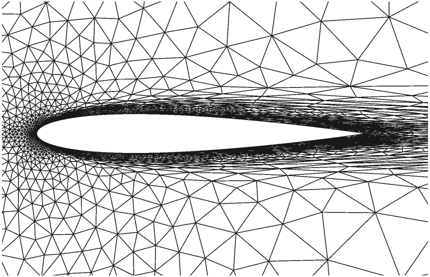


drag adapt (894 elems, ^{adapt})

Mesh Evolution



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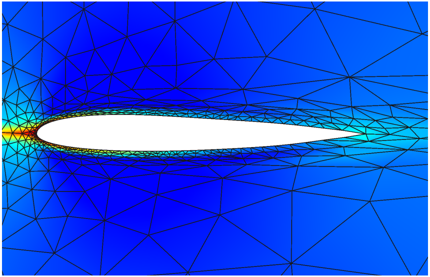
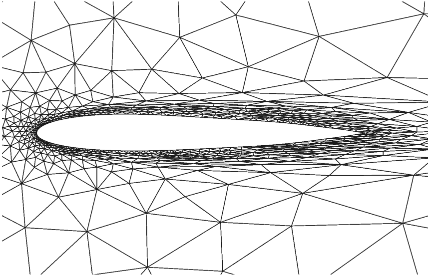
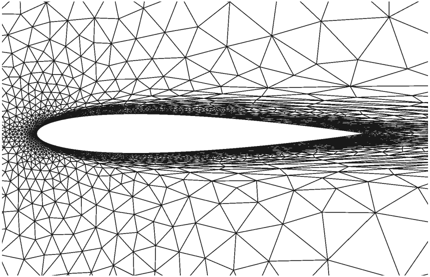


drag adapt (894 elems, $^{\text{adapt}}$)

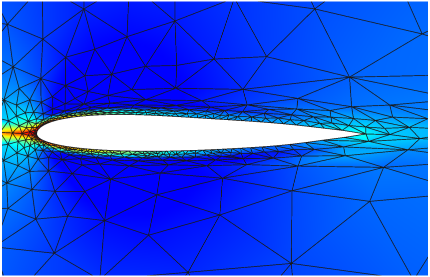


adapt equally (1253 elements)

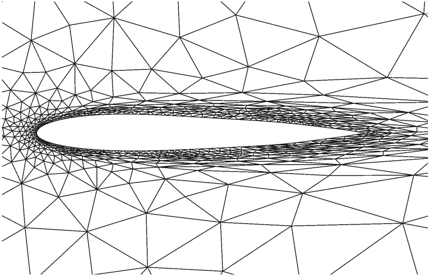
Mesh Evolution



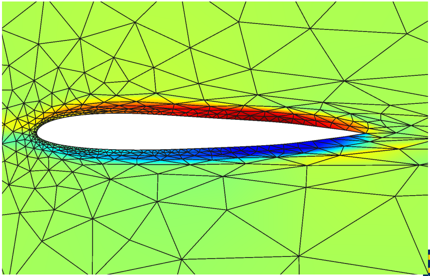
no adapt (3796 elements)



drag adapt (894 elems, $^{\text{adapt}}$)

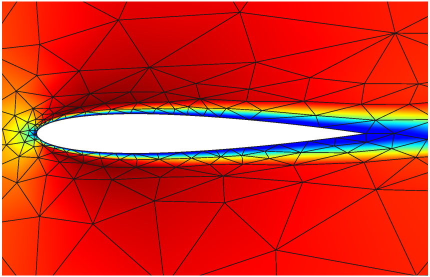


adapt equally (1253 elements)



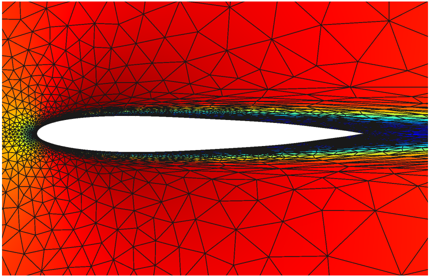
opt adapt (869 elems, $^{\text{trim}}$)

Optimized design (Mach Contour 0 0:6)



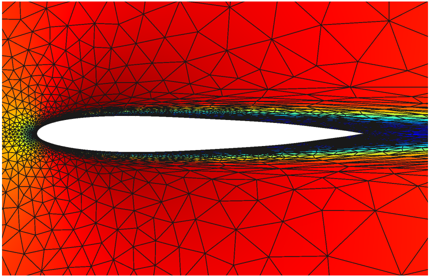
initial airfoil (= 0)

Optimized design (Mach Contour 0   0:6)

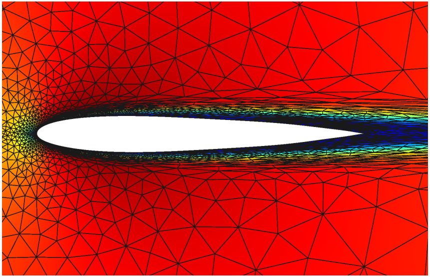


no adapt (= 0:27)

Optimized design (Mach Contour 0   0:6)

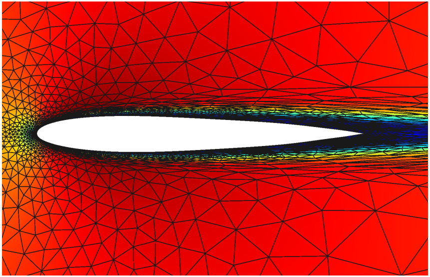


no adapt (= 0:27)

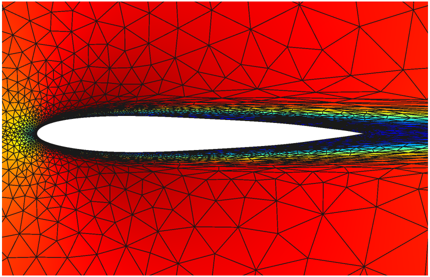


adapt on drag (= 0:30)

Optimized design (Mach Contour 0 0:6)



no adapt (= 0:27)



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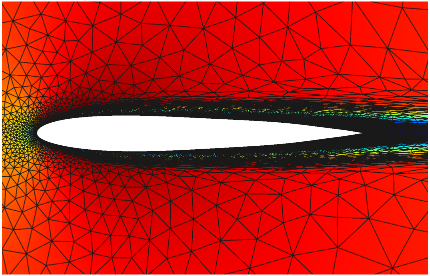
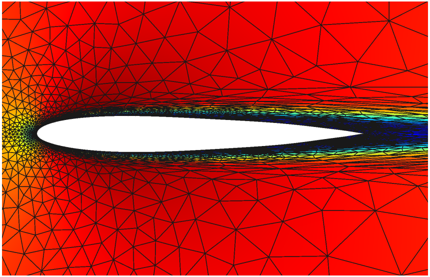
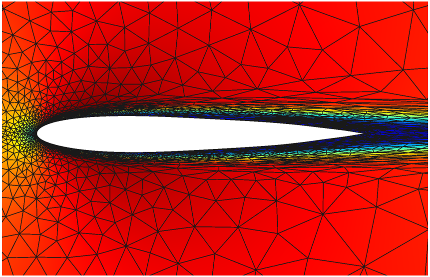


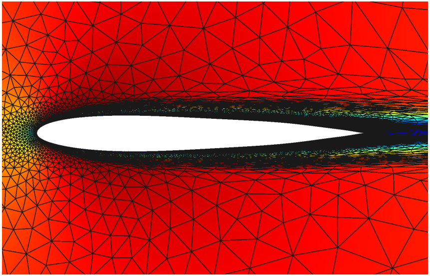
adapt equally (= 0:25)



no adapt (= 0:27)



adapt on drag (= 0:30)



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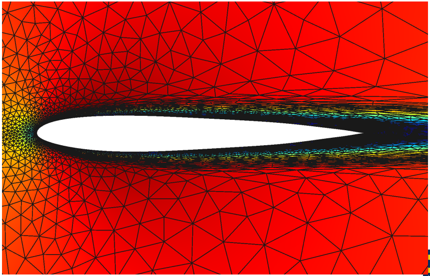
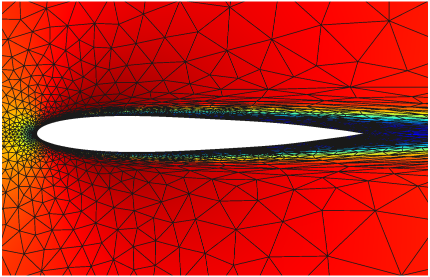
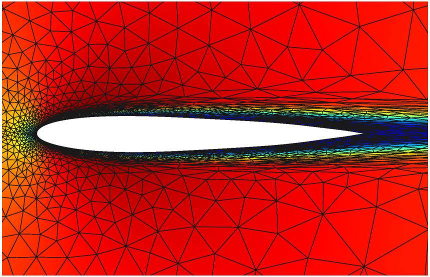


adapt on opt (= 0:24)

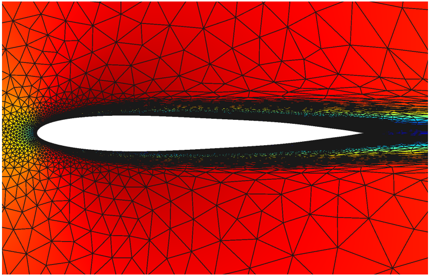
Optimized design (Mach Contour 0 0:6)



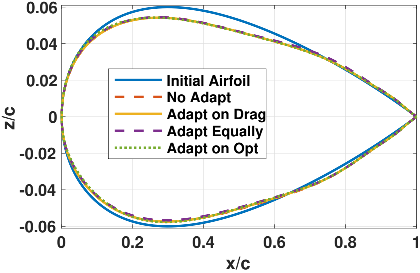
no adapt (= 0:27)



adapt on drag (= 0:30)



adapt equally (= 0:25)

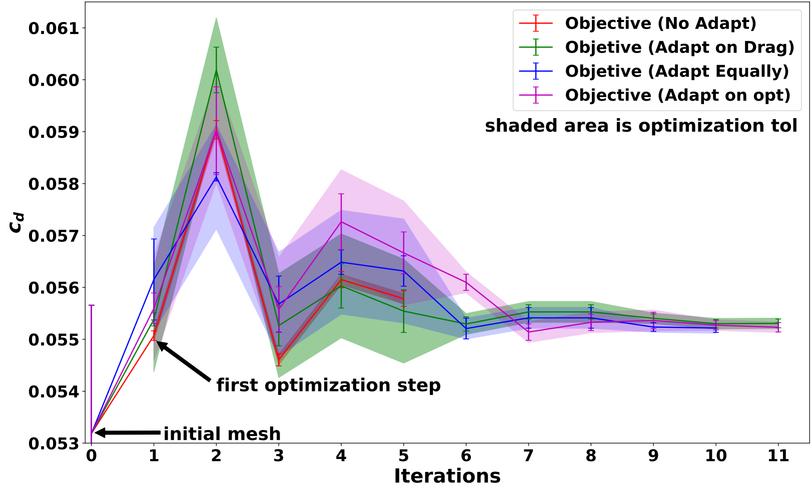


final design

Laminar, Subsonic Flow (infeasible starting point)

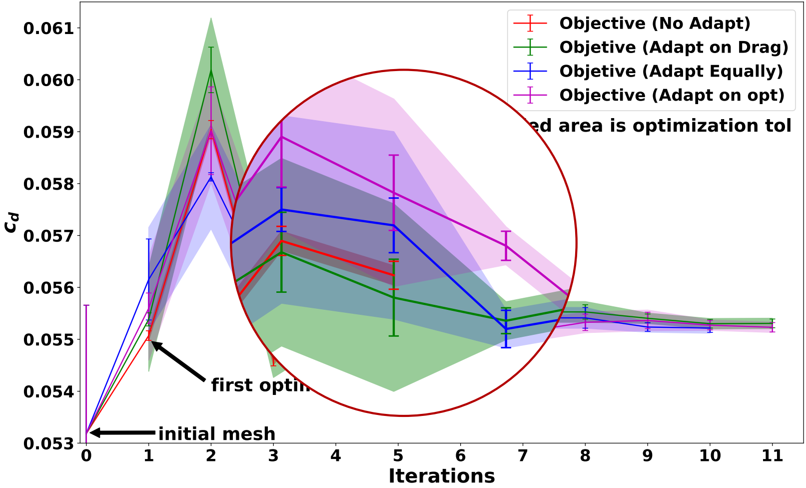
NACA 0012, $Re = 5000$, $M_{1} = 0{:}5$, $\;_0 = 0$

$J^{\text{adapt}} = c_d$, $J^{\text{trim}} = c_l$, $J^{\text{trim}} = 0{:}1$, $A \quad A_{min}$



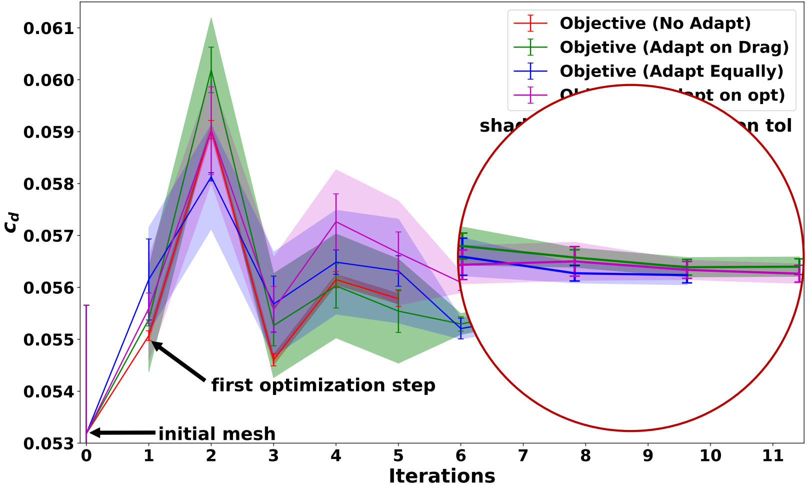
Laminar, Subsonic Flow (infeasible starting point)

NACA 0012, $Re = 5000$, $M_1 = 0{:}5$, $_0 = 0$
$J^{\text{adapt}} = c_d$, $J^{\text{trim}} = c_l$, $J^{\text{trim}} = 0{:}1$, $A \quad A_{min}$



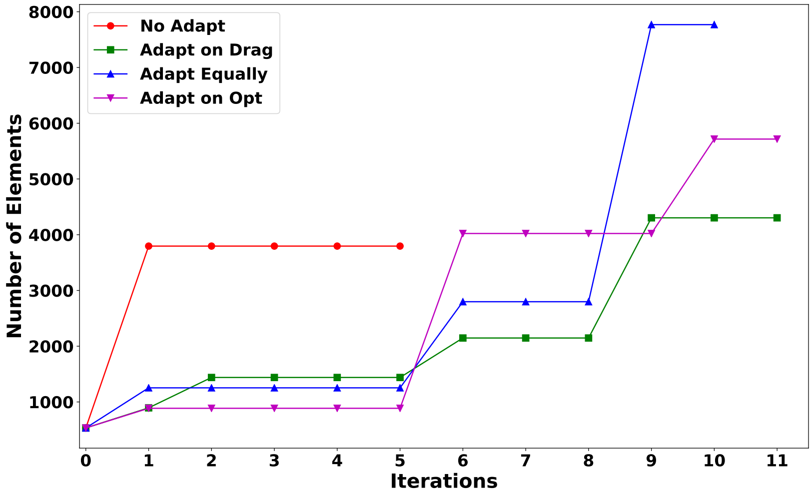
Laminar, Subsonic Flow (infeasible starting point)

NACA 0012, $Re = 5000$, $M_1 = 0{:}5$, $_0 = 0$
$J^{\text{adapt}} = c_d$, $J^{\text{trim}} = c_l$, $J^{\text{trim}} = 0{:}1$, $A \quad A_{min}$



Laminar, Subsonic Flow (infeasible starting point)

NACA 0012, $Re = 5000$, $M_1 = 0{:}5$, $_0 = 0$
$J^{\text{adapt}} = c_d$, $J^{\text{trim}} = c_l$, $J^{\text{trim}} = 0{:}1$, $A \quad A_{min}$



Mesh Evolution

initial mesh (533 elements)

Mesh Evolution

no adapt (3796 elements)

Mesh Evolution

no adapt (3796 elements)

drag adapt (1439 elems, ^{adapt})

Mesh Evolution

no adapt (3796 elements)

drag adapt (1439 elems, $^{\text{adapt}}$)

adapt equally (1253 elems)

Mesh Evolution

no adapt (3796 elements)

drag adapt (1439 elems, ^{adapt})

adapt equally (1253 elems)

opt adapt (886 elems, ^{trim})

Optimized design (Mach Contour 0 0:6)

initial airfoil (= 0)

Optimized design (Mach Contour 0   0:6)

no adapt (= 2:53)

Optimized design (Mach Contour 0 0:6)

no adapt (= 2:53)

adapt on drag (= 2:46)

Optimized design (Mach Contour 0   0:6)

no adapt (= 2:53)

adapt on drag (= 2:46)

adapt equally (= 2:39)

no adapt (= 2:53)

adapt on drag (= 2:46)

adapt equally (= 2:39)

adapt on opt (= 2:40)

Optimized design (Mach Contour 0   0:6)

no adapt (= 2:53)

adapt on drag (= 2:46)

adapt equally (= 2:39)

nal design

Inviscid, Transonic Flow

NACA 0012, $M_\infty = 0.8$, $\alpha_0 = 1.25$

$J^{adapt} = c_d$, $J^{trim} = c_l$, $J^{trim} = 0.4$, $A \geq A_{min}$

Turbulent, Low-Speed Flow

NACA 0012, $Re = 10^6$, $M_1 = 0{:}15$, $_0 = 6$
$J^{adapt} = c_d$, $J^{trim} = c_l$, $J^{trim} = 0{:}6$, $A \quad A_{min}$

initial mesh

rst step mesh

nal mesh

Turbulent, Low-Speed Flow

NACA 0012, $Re = 10^6$, $M_\infty = 0.15$, $\alpha_0 = 6$

$J^{adapt} = c_d$, $J^{trim} = c_l$, $J^{trim} = 0.6$, $A \geq A_{min}$

initial design | final design | airfoil shape

Outline

Conclusion

Conslusions:

- Numerical error should be carefully controlled as the shape and mesh change during the optimization
- We integrate output-based error estimation and mesh adaptation with a traditional gradient-based algorithm
- Error tolerance serves as the optimization tolerance at each delity, delity increases through mesh adaptation.
- Coupled adjoint error estimation offers a more ef cient way to adapt the mesh for constrained optimization problem
- Prevent over-re ning and over-optimizing during optimization

Future Work:

- Develop improved and automated delity increase strategy
- Avoid overshot re nement, allow more mesh redistribution and coarsening, optimize mesh during the optimization
- Combined with h-p re nement

Acknowledgments

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Boeing Company, with technical monitor Dr. Mori Mani
— Thank you —

Sensitivity Veri cation

Adapt Indicator

Complete Error Estimation

Error Estimation and Mesh Adaptation for Optimization

Flow problem:

$$
\begin{aligned}
J \quad & J_h(\mathbf{U}_h^H; \mathbf{x}) \quad J_h(\mathbf{U}_h; \mathbf{x}) \\
= \quad & {}_h^T \ \mathbf{R}_h \\
= \quad & {}_h^T \mathbf{R}_h(\mathbf{U}_h^H; \mathbf{x})
\end{aligned}
$$

Flow problem:

$$J \quad J_h(\mathbf{U}_h^H; \mathbf{x}) \quad J_h(\mathbf{U}_h; \mathbf{x})$$
$$= \quad {}_h^T \; \mathbf{R}_h$$
$$= \quad {}_h^T \mathbf{R}_h(\mathbf{U}_h^H; \mathbf{x})$$

During optimization:

Error Estimation and Mesh Adaptation for Optimization

Flow problem:

$$
\begin{aligned}
J \quad & J_h(\mathbf{U}_h^H;\mathbf{x}) \quad J_h(\mathbf{U}_h;\mathbf{x}) \\
= \quad & {}_h^T \; \mathbf{R}_h \\
= \quad & {}_h^T \mathbf{R}_h(\mathbf{U}_h^H;\mathbf{x})
\end{aligned}
$$

During optimization:

$$
\begin{aligned}
\frac{\partial L}{\partial \mathbf{x}} &= \frac{\partial J^{\text{adapt}}}{\partial \mathbf{x}} + {}^T\frac{\partial \mathbf{R}}{\partial \mathbf{x}} + {}^T\frac{\partial \mathbf{R}^{\text{trim}}}{\partial \mathbf{x}} \neq \mathbf{0} \\
\frac{\partial L}{\partial \mathbf{U}} &= \frac{\partial J^{\text{adapt}}}{\partial \mathbf{U}} + {}^T\frac{\partial \mathbf{R}}{\partial \mathbf{U}} + {}^T\frac{\partial \mathbf{R}^{\text{trim}}}{\partial \mathbf{U}} = \mathbf{0} \\
\frac{\partial L}{\partial} &= \mathbf{R}(\mathbf{U};\mathbf{x}) = \mathbf{0} \\
\frac{\partial L}{\partial} &= \mathbf{R}^{\text{trim}}(\mathbf{U};\mathbf{x}) \neq \mathbf{0}
\end{aligned}
$$

Error Estimation and Mesh Adaptation for Optimization

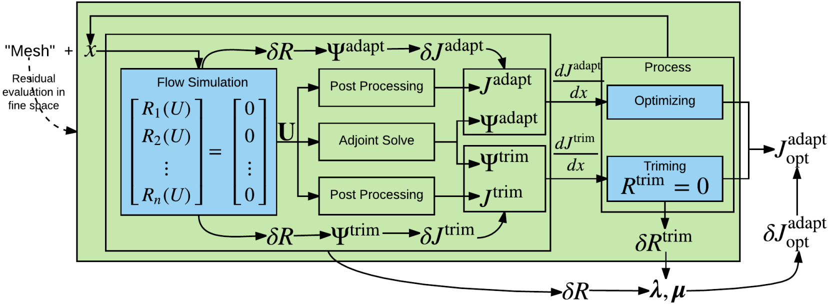
Flow problem:

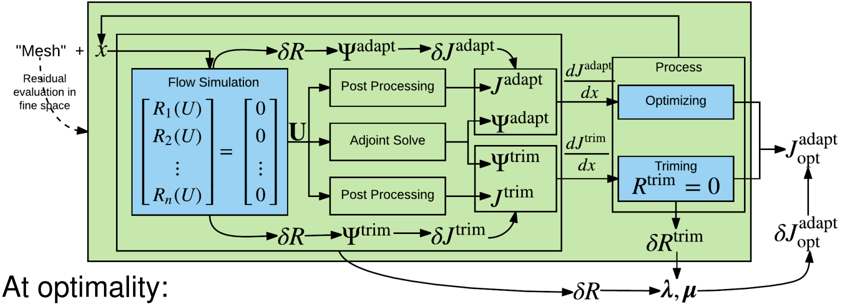
$$J \quad J_h(\mathbf{U}_h^H;\mathbf{x}) \quad J_h(\mathbf{U}_h;\mathbf{x})$$
$$= \quad {}_h^T \; \mathbf{R}_h$$
$$= \quad {}_h^T \mathbf{R}_h(\mathbf{U}_h^H;\mathbf{x})$$

During optimization:

$$J^{\text{adapt}} = \quad {}_h^T \; \mathbf{R}_h \quad {}^T \; \mathbf{R}_h^{\text{trim}}$$
$$= \quad {}^T\mathbf{R}_h(\mathbf{U}_h^H;\mathbf{x}) \quad {}^T\mathbf{R}_h^{\text{trim}}(\mathbf{U}_h^H;\mathbf{x})$$
$$= \left({}_h^{\text{adapt}} + \quad {}_h \quad {}_h \right)^T \mathbf{R}_h(\mathbf{U}_h^H;\mathbf{x}) \quad {}_h^T (J_h^{\text{trim}}(\mathbf{U}_h^H;\mathbf{x}) \quad J_h^{\text{trim}}(\mathbf{U}_h;\mathbf{x}))$$
$$= J^{\text{adapt}} + {}_h^T \underbrace{(\quad {}_h^{\text{trim}})^T \mathbf{R}_h(\mathbf{U}_h^H;\mathbf{x})}_{J^{\text{trim}}} \quad {}_h^T \; J^{\text{trim}}$$
$$= \quad J^{\text{adapt}}$$

Error Estimation and Mesh Adaptation for Optimization





At optimality:

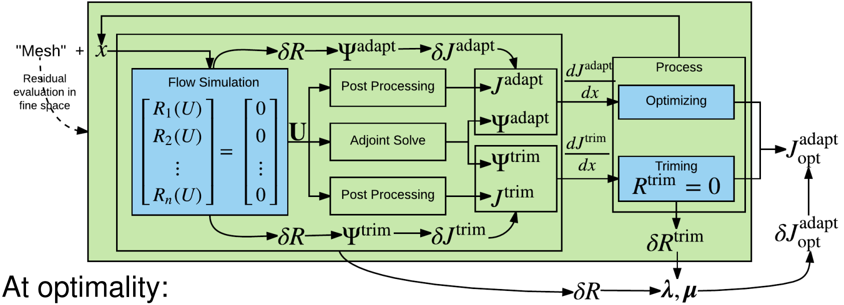
$$\frac{\partial L}{\partial \mathbf{x}} = \frac{\partial J^{\text{adapt}}}{\partial \mathbf{x}} + {}^T\frac{\partial \mathbf{R}}{\partial \mathbf{x}} + {}^T\frac{\partial \mathbf{R}^{\text{trim}}}{\partial \mathbf{x}} = \mathbf{0}$$

$$\frac{\partial L}{\partial \mathbf{U}} = \frac{\partial J^{\text{adapt}}}{\partial \mathbf{U}} + {}^T\frac{\partial \mathbf{R}}{\partial \mathbf{U}} + {}^T\frac{\partial \mathbf{R}^{\text{trim}}}{\partial \mathbf{U}} = \mathbf{0}$$

$$\frac{\partial L}{\partial} = \mathbf{R}(\mathbf{U}; \mathbf{x}) = \mathbf{0}$$

$$\frac{\partial L}{\partial} = \mathbf{R}^{\text{trim}}(\mathbf{U}; \mathbf{x}) = \mathbf{0}$$

Error Estimation and Mesh Adaptation for Optimization



At optimality:

$$J^{adapt}_{opt} = \quad {}^{T}_{h}\ \mathbf{R}_h \qquad {}^{T}\ \mathbf{R}^{trim}_h$$

$$= \quad {}^{T}\mathbf{R}_h(\mathbf{U}^H_h; \mathbf{x}_H) \qquad {}^{T}\mathbf{R}^{trim}_h(\mathbf{U}^H_h; \mathbf{x}_H)$$

$$= \quad (\ \ {}^{adapt}_{h} + \quad {}_{h}\ \ {}_{h})^T\mathbf{R}_h(\mathbf{U}^H_h; \mathbf{x}_H) \qquad {}^{T}_{h}\underbrace{(J^{trim}_h(\mathbf{U}^H_h; \mathbf{x}_H) \qquad J^{trim})}_{0}$$

$$= \quad J^{adapt}(\mathbf{x}_H) + \quad {}^{T}_{h}\ J^{trim}(\mathbf{x}_H)$$

Note: $J^{trim}_h(\mathbf{U}^H_h; \mathbf{x}_H) = J^{trim}_H(\mathbf{U}_H; \mathbf{x}_H) = J^{trim} = J^{trim}_h(\mathbf{U}_h; \mathbf{x}_h)$

Error Estimation and Mesh Adaptation for Optimization

Optimality error estimation

$$\frac{dJ^{\text{adapt}}}{d\mathbf{x}} + {}^{T}\frac{d\mathbf{J}^{\text{trim}}}{d\mathbf{x}} = \mathbf{0}$$

$$J^{\text{adapt}}_{\text{opt}} = J^{\text{adapt}}(\mathbf{x}_H) + {}^{T}_{h}\ J^{\text{trim}}(\mathbf{x}_H)$$

What does mean?

It is the objective sensitivity w.r.t constraints and measures how much the constraints error can affect the optimal objective.

Mesh adaptation implementation

Adapt (error) indicator: $= | \ {}^{T}_{h;}\ \mathbf{R}_{h;}\ (\mathbf{U}^{H}_{h}; \mathbf{x}_H)|$

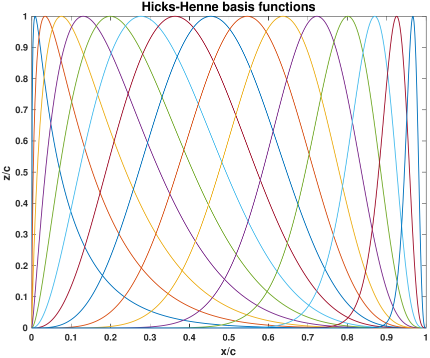
Combined indicator: ${}_{;\text{opt}} = \underbrace{\ {}^{\text{adapt}}}_{\text{objective error only}} + \underbrace{|\ |^{T}\ {}^{\text{trim}}}_{\text{inexact constraints satisfaction}}$

${}_h$: reconstructing the coarse-space adjoints ${}_H$

${}_h$: extracted from the optimizer on the coarse space

Airfoil Parameterization

- Hicks-Henne basis functions: linear combination of "bump" functions added to the baseline airfoil



$$z = z_{\text{base}} + \sum_{i=0}^{n} a_i \;\; {}_i(x)$$

$${}_i(x) = \sin^{t_i}(\;\; x^{m_i})$$

$$m_i = \ln(0.5)/\ln(x_{M_i})$$

x : coord along the airfoil chord
z : vertical surface coord
x_{M_i} : maxima location
t_i : width of the bump function

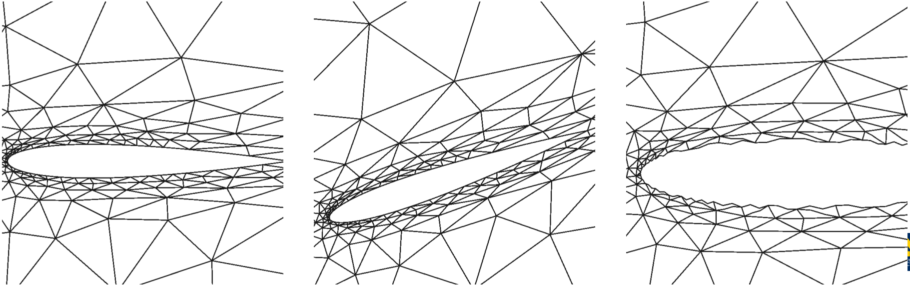
- Design parameters: $\mathbf{x} = [\;\; ; a_1 ; a_2 ; \;\;\; ; a_n]^T$
  Coefficients of Hicks-Henne basis + angle of attack

Mesh Movement

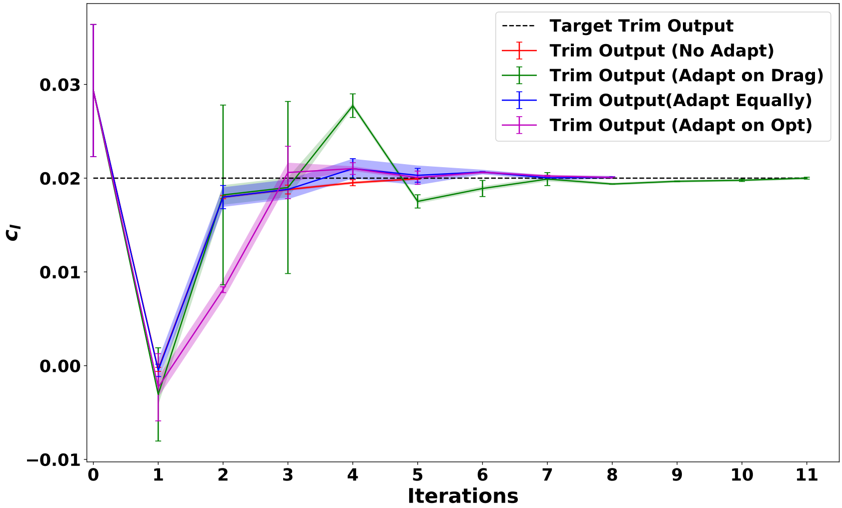
- Radial Basis Function (RBF): only depends on the distance from the origin or a center, e.g. $(x) = e^{\ x^2}$
- We can use a sum of RBFs $(\|\bar{x}\|)$ and a polynomial $p(\bar{x})$ to interpolate the original function (mesh movement):

$$\tilde{d}(\bar{x}) \quad \tilde{d}(\bar{x}) = \sum_{i=1}^{N_b} \bar{r}_i \ (\|\bar{x} \quad \bar{x}_i\|) + \bar{p}(\bar{x})$$

- Solving for a linear system $O(N_b)$ of $\bar{r}_i$
- Mesh connectivity information not required



Constraints, Nearly Feasible Starting Point



Constraints, Infeasible Starting Point